

LINEAR RECURRENCE RELATIONS FOR BINOMIAL COEFFICIENTS MODULO A PRIME

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ABSTRACT. We investigate when the sequence of binomial coefficients $\binom{k}{i}$ modulo a prime p , for a fixed positive integer k , satisfies a linear recurrence relation of (positive) degree h in the finite range $0 \leq i \leq k$. In particular, we prove that this cannot occur if $2h \leq k < p - h$. This hypothesis can be weakened to $2h \leq k < p$ if we assume, in addition, that the characteristic polynomial of the relation does not have -1 as a root. We apply our results to recover a known bound for the number of points of a Fermat curve over a finite field.

1. INTRODUCTION

As is customary, let the binomial coefficients $\binom{k}{i}$ be defined by the identity

$$(1+x)^k = \sum_{i \in \mathbb{Z}} \binom{k}{i} x^i = \sum_{i \geq 0} \binom{k}{i} x^i$$

in the ring of formal power series $\mathbb{Z}[[x]]$, for $k, i \in \mathbb{Z}$. In particular, $\binom{k}{i}$ vanishes if $i < 0$ or if $i > k$. Consider the sequence $\binom{k}{i}$ for a fixed k . It is clearly never periodic on the whole range $i \in \mathbb{Z}$, and restricted to the range $i \geq 0$ it is periodic exactly in one case, namely $k = -1$, where $\binom{k}{i} = (-1)^i$. Replicas of this isolated instance in characteristic zero appear when the sequence $\binom{k}{i}$ is viewed modulo a prime p : the reduced sequence is periodic with period two in the natural range $0 \leq i \leq k$ if $k+1$ is a power of p . In fact, because of the identity $(a+b)^p = a^p + b^p$ in characteristic p , in the ring $\mathbb{F}_p[x]$ we have

$$(1+x)^{p^s-1} = (1+x^{p^s})/(1+x) = (1+x^{p^s}) \sum_{i \geq 0} (-1)^i x^i = \sum_{i=0}^{p^s-1} (-1)^i x^i$$

and hence $\binom{p^s-1}{i} \equiv (-1)^i \pmod{p}$ for $0 \leq i \leq k$. We have proved in [Matb] that under some fairly natural further assumptions this is the only occurrence of periodicity for the sequence $\binom{k}{i}$ modulo p in the range $0 \leq i \leq k$, for a fixed $k \geq 0$. In particular, Corollary 4.2 of [Matb] asserts that if $k+1$ is not a power of p then the sequence of binomial coefficients $\binom{k}{i}$ modulo p , considered in the range $0 \leq i \leq k$, cannot be periodic of any period h prime to p and with $2h \leq k$. A similar assertion holds for the signed binomial coefficients $(-1)^i \binom{k}{i}$. In fact, both assertions hold under weaker and more precise assumptions, for which we refer to [Matb].

Since a periodicity relation is the special case of a linear recurrence relation where the characteristic polynomial has the form $x^h - 1$, it is natural to ask when the sequence $\binom{k}{i}$ modulo p , for k fixed, satisfies a linear recurrence relation in the range

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$0 \leq i \leq k$. Note that, in characteristic zero and in the range $i \geq 0$, the sequence $\binom{k}{i}$ satisfies the linear recursion with characteristic polynomial $(1+x)^{-k}$ when $k < 0$ (see Example 8 for a similar instance with $k \geq 0$ in positive characteristic), and the linear recursion with characteristic polynomial x^{k+1} for $k \geq 0$ (because the sequence vanishes for $i > k$). However, the problem becomes more interesting when we restrict our attention to the finite range $0 \leq i \leq k$, for $k \geq 0$. Naturally, the familiar definitions pertaining to linear recurrence relations for infinite sequences need to be adjusted to the case of finite sequences, as we do in Section 2. In particular, it will appear that a natural requirement to avoid degenerate cases is to consider linear recurrence relations of order h only for sequences of at least $2h+1$ terms, see Remark 3 and Example 7. In the case of the sequence $\binom{k}{i}$ modulo p in the natural range $0 \leq i \leq k$, this assumption becomes $2h \leq k$, which we have already encountered in Corollary 4.2 of [Matb] quoted above. An analogue of that result for linear recurrence instead of periodicity is our Theorem 6. A simplified version of that asserts that the sequence of binomial coefficients $\binom{k}{i}$ modulo p restricted to the range $0 \leq i \leq k$ cannot satisfy a linear recurrence relation of degree h if $0 < 2h \leq k < p-h$. The assumption $k < p-h$, which is indispensable according to Example 8, can be weakened to $k < p$ provided we assume that the characteristic polynomial of the linear recurrence relation does not have -1 as a root, as in Theorem 10.

We base our proofs of Theorems 6 and 10 on two different methods. Both methods would actually work in both cases, as we explain in Remark 11, but each method may have its own strengths in view of possible generalizations, notably to values of k larger than p . The first method comes naturally from the ordinary theory of linear recurrent sequences and consists in evaluating certain *Hankel determinants*. We do that in Proposition 4, which may be of independent interest. The second method employs the generating function $(1+x)^k$ for the binomial coefficients in a more explicit way. It is based on Lemma 9, a slight extension of an elementary fact taken from [HBK00], asserting that the number of nonzero coefficients of a polynomial in characteristic p exceeds the multiplicity of one of its roots, provided that multiplicity is less than p . The characteristic zero analogue of this fact appears as Lemma 1 in [Bri01], but may be well known.

The same arguments employed to prove Theorems 6 and 10 can be used to recover a known bound for the number of points of a Fermat curve on a finite field. We do this in Section 4, to which we refer for an introduction to the problem. Since this topic does not require an understanding of linear recurrence relations, we have kept Section 4 essentially independent from the rest of the paper. However, we do explain the connection with linear recurrence relations for binomial coefficients in Remark 13.

2. LINEAR RECURRENCE RELATIONS OVER A FINITE RANGE

In this section we recall from [LN86] some basic concepts concerning (homogeneous) linear recurrences satisfied by an infinite sequence, and adapt them to finite sequences.

Let $a(x) = \sum_{i \geq 0} a_i x^i$ be a monic polynomial of degree h (possibly zero). A sequence $\{s_i\}_{i \geq 0}$ satisfies the (homogeneous) linear recurrence relation with *characteristic polynomial* $a(x)$ if

$$(1) \quad s_{i+h} = -a_{h-1}s_{i+h-1} - \cdots - a_0 s_i \quad \text{for all } i \geq 0.$$

The degree h of $a(x)$ is called the *order* of the linear recurrence relation, in analogy with the terminology used for linear differential equations. This definition of a linear recurrence relation is motivated by applications where the recurrence allows

one to compute s_i from the h elements preceding it in the sequence. However, the *reciprocal characteristic polynomial* $a^*(x) = x^h a(1/x) = \sum_{i \geq 0} a_{h-i} x^i$ is somehow more suited to algebraic manipulations than the characteristic polynomial. (Note that $a^*(x)$ may have degree lower than $a(x)$, namely, when $a_0 = 0$.) In particular, setting $a_i^* = a_{h-i}$, and hence $a^*(x) = \sum_{i \geq 0} a_i^* x^i$, we can rewrite (1) as

$$a_h^* s_i + a_{h-1}^* s_{i+1} + \cdots + a_0^* s_{i+h} = 0 \quad \text{for all } i \geq 0.$$

These equations impose the vanishing of the coefficient of x^{i+h} in the product $a^*(x)s(x)$, for all $i \geq 0$, where $s(x) = \sum_{i \geq 0} s_i x^i$ is the *generating function* of the sequence $\{s_i\}_{i \geq 0}$. Therefore, a sequence $\{s_i\}_{i \geq 0}$ satisfies the linear recurrence associated with $a(x)$ if and only if $a^*(x)s(x)$ is a polynomial of degree less than h (cf. [LN86, Theorem 6.40]).

Consider now a finite sequence $\{s_i\}_{u \leq i \leq v}$, and let $s(x) = \sum_{u \leq i \leq v} s_i x^i$ be its generating function. We may encompass the classical case of infinite sequences by allowing $v = \infty$. Since we are not assuming that $u \geq 0$, in general $s(x)$ is a formal Laurent series rather than an ordinary power series.

Definition 1. Let $a(x) = \sum_{i \geq 0} a_i x^i$ be a monic polynomial of degree h . The sequence $\{s_i\}_{u \leq i \leq v}$ satisfies the linear recurrence with *characteristic polynomial* $a(x)$ if

$$(2) \quad a_0 s_i + a_1 s_{i+1} + \cdots + a_h s_{i+h} = 0 \quad \text{for } u \leq i \leq v - h.$$

A finite sequence $\{s_i\}_{u \leq i \leq v}$ satisfies the linear recurrence with characteristic polynomial $a(x)$ if and only if the sequence can be extended to an infinite sequence $\{s_i\}_{i \geq u}$ satisfying the linear recurrence in the usual sense. This reveals a slight asymmetry in Definition 1 with respect to reversing the ordering of the finite sequence, due to our requirement that a_h be nonzero (and then, without loss, being equal to one) without a similar requirement on a_0 . We accept to live with this harmless asymmetry rather than departing from the standard terminology used for infinite sequences.

Let $a^*(x) = x^h a(1/x) = \sum_{i \geq 0} a_{h-i} x^i$ be the *reciprocal characteristic polynomial*. Arguing as in an earlier paragraph we find that (2) is satisfied if and only if the coefficient of x^j in the polynomial $a^*(x)s(x)$ vanishes for $u + h \leq j \leq v$. In particular, a finite sequence $\{s_i\}_{u \leq i \leq v}$ satisfies vacuously any linear recurrence relation of order $h > v - u$. It also follows easily that if the sequence satisfies a linear recurrence relation with characteristic polynomial $a(x)$ then it satisfies any linear recurrence relation whose characteristic polynomial is a multiple of $a(x)$.

The *Hankel determinants* $D_r^{(h)} = \det((s_{r+i+j})_{i,j=0,\dots,h-1})$ play an important role in the ordinary theory of infinite linear recurring sequences, see [LN86, Chapter 6]. With some care some of their properties can be translated to the present setting of finite sequences. Here we limit ourselves to the following basic fact.

Lemma 2. *If the sequence $\{s_i\}_{u \leq i \leq v}$ satisfies a linear recurrence of order h then $D_r^{(h+1)} = 0$ for $u \leq r \leq v - 2h$.*

Proof. View (2) as a system of $v - h - u + 1$ homogeneous linear equations in the $h + 1$ indeterminates a_0, \dots, a_h . The Hankel determinants $D_r^{(h+1)}$ under consideration are the determinants of the subsystems consisting of $h + 1$ consecutive equations. The existence of a nonzero solution (with $a_h = 1$ here) implies that the matrix of the system has rank less than $h + 1$, and hence all Hankel determinants vanish. \square

Note that the values for r in Lemma 2 are all those for which $D_r^{(h+1)}$ is defined. In particular, the conclusion of Lemma 2 is void if $2h > v - u$.

Remark 3. Given a finite sequence $\{s_i\}_{u \leq i \leq v}$ and a positive integer h with $h \leq v - u < 2h$, the set of $v - h - u + 1$ equations given by (2) necessarily has a nonzero solution (a_0, \dots, a_h) , but need not have any with $a_h \neq 0$. In fact, the sequence need not satisfy any linear recurrence relation of order h in this case, as is shown by the sequence with $s_v = 1$ and $s_i = 0$ for $u \leq i < v$. Nevertheless, the condition $2h \leq v - u$ is a natural assumption when claiming that a finite sequence *does not* satisfy a linear recurrence relation, as in Theorems 6 and 10 below. We examine a specific instance in Example 7.

3. LINEAR RECURRENCE RELATIONS FOR BINOMIAL COEFFICIENTS

One way of studying linear recurrence relations for the sequence of binomial coefficients $\binom{k}{i}$ (for k fixed) is evaluating the corresponding Hankel determinants. We use the notation $k^{\underline{r}} = k(k-1) \cdots (k-r+1)$ for k and r integers with $r \geq 0$, reading $k^{\underline{0}} = 1$.

Proposition 4. *Let k be an integer and let h, r be nonnegative integers. Let $B(k, h, r)$ denote the matrix $\left(\binom{k}{r+i+j}\right)_{i,j=0,\dots,h}$. Then we have*

$$\det(B(k, h, r)) = (-1)^{\binom{h+1}{2}} \prod_{s=0}^h \frac{(k+s)^{\overline{r+h}}}{(r+h+s)^{\overline{r+h}}}.$$

Proof. We proceed by induction on h . The conclusion holds for $h = 0$ since $\binom{k}{r} = \frac{k^{\underline{r}}}{r^{\underline{r}}}$. Now assume $h > 0$. To avoid confusion, we count the rows and columns of a matrix according to their index. Thus, we call 0th row the earliest row of $B(k, h, r)$. We compute the determinant according to Laplace's rule, with respect to the last column (that is, the h th column). For $i = h, h-1, \dots, 1$ (in this order) we subtract from the i th row the $(h-1)$ st row multiplied by $\frac{k+1}{r+h+i} - 1$. Using the binomial identities $\binom{a}{b+1} + \binom{a}{b} = \binom{a+1}{b+1}$ and $\binom{a}{b} \frac{a+1}{b+1} = \binom{a+1}{b+1}$ we find that the (i, j) -entry of the resulting matrix, for $i > 0$, equals

$$\begin{aligned} & \left(\binom{k}{r+i+j} - \binom{k}{r+i+j-1} \left(\frac{k+1}{r+h+i} - 1 \right) \right) \\ &= \binom{k+1}{r+i+j} - \binom{k}{r+i+j-1} \frac{k+1}{r+h+i} \\ &= \binom{k+1}{r+i+j} \left(1 - \frac{r+i+j}{r+h+i} \right) = \binom{k+1}{r+i+j} \frac{h-j}{r+h+i}. \end{aligned}$$

In particular, the h -th column of $B(k, h, r)$ vanishes except for its $(0, h)$ -entry, which equals $\binom{k}{r+h}$. Consequently, $\det(B(k, h, r))$ equals $(-1)^h \binom{k}{r+h}$ times the determinant of the matrix obtained by removing from it the 0th row and the h th column. After shifting its row-index by one, the latter matrix becomes

$$(3) \quad \left(\binom{k+1}{r+1+i+j} \frac{h-j}{r+h+i+1} \right)_{i,j=0,\dots,h-1}.$$

By collecting the factor $1/(r+h+i+1)$ from the i th row and the factor $h-j$ from the j th column, for each row and column, we find that the determinant of the matrix in (3) equals the product of $\binom{r+2h}{h}^{-1}$ and $\det(B(k+1, h-1, r+1))$. Since $\binom{k}{r+h} \binom{r+2h}{h}^{-1} = \frac{k^{\overline{r+h}}}{(r+2h)^{\overline{r+h}}}$ we conclude that

$$\det(B(k, h, r)) = (-1)^h \frac{k^{\overline{r+h}}}{(r+2h)^{\overline{r+h}}} \cdot \det(B(k+1, h-1, r+1)).$$

By induction hypothesis we have

$$\begin{aligned} \det(B(k, h, r)) &= (-1)^{h+\binom{h}{2}} \frac{k^{r+h}}{(r+2h)^{r+h}} \prod_{s=0}^{h-1} \frac{(k+1+s)^{r+h}}{(r+h+s)^{r+h}} \\ &= (-1)^{h+\binom{h}{2}} \prod_{s=0}^h \frac{(k+s)^{r+h}}{(r+h+s)^{r+h}}, \end{aligned}$$

which concludes the proof. \square

Corollary 5. *Let p be a prime and let k, h, r be integers with $k, h \geq 0$ and $0 \leq r+h \leq k < p-h$. Then $\det(B(k, h, r))$ is not a multiple of p .*

Proof. The conclusion follows from Proposition 4 if $r \geq 0$. The case where $r < 0$ is reduced to the other case by means of the identity

$$\det(B(k, h, r)) = \det(B(k, h, k-r-2h)),$$

which follows from the identity $\binom{k}{i} = \binom{k}{k-i}$ for the binomial coefficients. \square

The necessity of the conditions $0 \leq r+h \leq k$ in Corollary 5 can also be seen by noting that the matrix $B(k, h, -h-1)$ (respectively $B(k, h, k-h+1)$) has zeroes on and above (respectively below) its secondary diagonal.

Our first main result states that the sequence of binomial coefficients $\binom{k}{i}$, considered in the range $i = 0, \dots, k$ and reduced modulo a prime p , does not satisfy any recurrence relation of degree h if $2h \leq k < p-h$. However, in view of an application in the next section it is convenient to allow a more general range for i .

Theorem 6. *Let p be a prime, k, h nonnegative integers, and u, v integers with*

$$-h \leq u \leq v \leq k+h, \quad 2h \leq v-u, \quad \text{and} \quad k < p-h.$$

Then the sequence of binomial coefficients $\binom{k}{i}$ modulo p , considered in the range $u \leq i \leq v$, does not satisfy any linear recurrence relation of order h .

Proof. Suppose for a contradiction that the sequence under consideration satisfies a linear recurrence relation of order h . According to Lemma 2, the Hankel determinants $D_r^{(h+1)}$ of the sequence vanish for $u \leq r \leq v-2h$. Since $D_r^{(h+1)}$ is the reduction modulo p of $\det(B(k, h, r))$, it follows in particular that p divides $\det(B(k, h, u))$. This contradicts Corollary 5. \square

The extreme case $h = 0$ of Theorem 6 amounts to the simple fact that p does not divide $\binom{k}{i}$ for $0 \leq i \leq k < p$. Corollary 5 and Theorem 6 can be interpreted in characteristic zero by reading $p = \infty$ (or, equivalently, by disregarding the hypotheses which involve p).

We present a couple of examples to justify the hypotheses $2h+u+v \leq k$ and $k < p-h$ in Theorem 6, which are of a quite different nature.

Example 7. Work in characteristic zero first, and let k, h be integers with $0 < 2h-1 = k$. Then the sequence of binomial coefficients $\binom{k}{i}$, considered in the range $0 \leq i \leq k$, satisfies a unique linear recurrence relation of order h . In fact, we may set $a_h = 1$ and view (2) as a system of $k+1 = 2h$ linear equations in the h indeterminates a_0, \dots, a_{h-1} . Since its matrix $B(k, h-1, 0)$ is nonsingular according to Proposition 4, the system has a unique solution, which can be computed by means of the Berlekamp-Massey algorithm described in [LN86, Chapter 6, §6]. Consequently, the reduction of the sequence modulo any prime p also satisfies a linear recurrence, and this is unique if $p \geq k+h$ according to Corollary 5.

Example 8. If q is a power of the prime p and $0 < h \leq q$ then the sequence of binomial coefficients $\binom{q-h}{i}$ modulo p , considered in the range $-h+1 \leq i \leq q-1$ (which includes the more natural range $0 \leq i \leq q-h$), satisfies the linear recurrence relation with (reciprocal) characteristic polynomial $(1+x)^h$. This is so because the product of its generating function $(1+x)^{q-h}$ with $(1+x)^h$ equals $1+x^q$, whose coefficients of degrees $1, \dots, q-1$ vanish.

As Example 8 suggests, it is possible to weaken the hypothesis that $k < p-h$ in Theorem 6 to $k < p$ provided one assumes that the characteristic polynomial of the linear recurrence relation does not have 1 as root. This can be established by a variation of the method of proof of Theorem 6, which we sketch in Remark 11 below. However, it is simpler to base a proof on a different method. We need the following refinement of Lemma 6 of [HBK00], which had the stronger hypothesis $\deg(f) < p$. We call *weight* of a polynomial the number of its nonzero coefficients.

Lemma 9. *Let $f(x)$ be a polynomial over a field of characteristic p having a nonzero root ξ with multiplicity exactly k , with $0 < k < p$. Then $f(x)$ has weight at least $k+1$.*

Proof. If ξ is a root of $f(x)$ with multiplicity exactly k , then $f(\xi^{-1}x)$ has 1 as a root with the same multiplicity, and has the same weight as $f(x)$. Hence we may assume that $\xi = 1$. We proceed by induction on k . The case $k = 1$ being obvious, assume that $k > 1$. By dividing $f(x) = \sum_i f_i x^i$ by a suitable power of x , which leaves its weight unchanged, we may assume that $f_0 \neq 0$. Since p does not divide k , the derivative

$$f'(x) = \sum_i i f_i x^{i-1}$$

has 1 as a root with multiplicity exactly $k-1$, and has weight one less than the weight of $f(x)$. By induction, $f'(x)$ has weight at least k , and hence $f(x)$ has weight at least $k+1$. \square

Theorem 10. *Let p be a prime, k, h nonnegative integers, and u, v integers with*

$$-h \leq u \leq v \leq k+h, \quad 2h \leq v-u, \quad \text{and} \quad k < p.$$

Then the sequence of binomial coefficients $\binom{k}{i}$ modulo p , considered in the range $u \leq i \leq v$, does not satisfy any linear recurrence relation of order h with characteristic polynomial prime to $x+1$.

Proof. Suppose that the sequence under consideration satisfies a linear recurrence relation with characteristic polynomial $a(x)$, of degree h and not having -1 as a root. Therefore, the coefficient of x^j in the polynomial $a^*(x)(1+x)^k$ vanishes for $u+h \leq j \leq v$. Consequently, $a^*(x)(1+x)^k$ has weight at most $k+2h-v+u$. However, according to Lemma 9 the weight of $a^*(x)(1+x)^k$ is at least $k+1$. It follows that $2h \geq v-u+1$, which contradicts one of our hypotheses. \square

Remark 11. We mentioned above that the method of proof of Theorem 6 can be adapted to give a proof of Theorem 10. We briefly sketch the corresponding argument. Assume that $p-h \leq k < p$, otherwise Theorem 6 applies. As in the proof of Lemma 2 one may view (2) as a system of $v-h-u+1$ homogeneous linear equations in the $h+1$ indeterminates a_0, \dots, a_h . The matrix of the subsystem formed by the first $h+1$ equations equals the reduction modulo p of $B(k, h, u)$, whose determinant vanishes according to Corollary 5. However, one can use Corollary 5 to show that the matrix has rank at least $p-k$. (It will turn out that the matrix has rank exactly $p-k$.) Therefore, the space of solutions of the system (2) has dimension at least $h+1-p+k$. According to Example 8 and an earlier observation, the sequence under consideration satisfies any linear recurrence relation which has a

multiple of $(1+x)^{p-k}$ as characteristic polynomial. Those characteristic polynomials (assumed monic here) which have degree h form an affine subspace of $\mathbb{F}_p[x]$ of dimension $h - p + k$, and hence span a linear subspace of dimension $h - p + k + 1$. It follows that these account for all solutions of the system (2), which is the desired conclusion.

It is also possible to use Lemma 9 to prove Theorem 6. In fact, the weaker version of Lemma 9 given in [HBK00], where $\deg(f) < p$, would suffice for that.

4. AN APPLICATION TO FERMAT CURVES OVER A FINITE FIELD

Let \mathbb{F}_q be the finite field of q elements and let p be its characteristic. Consider the *Fermat curve* $ax^n + by^n = z^n$, expressed in homogeneous coordinates, where $n > 1$ is an integer prime to p , and $a, b \in \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. A classical estimate on the number $N_n(a, b, q)$ of its projective \mathbb{F}_q -rational points is

$$|N_n(a, b, q) - q - 1| \leq (n-1)(n-2)\sqrt{q}.$$

This is originally due to Hasse and Davenport [DH35] but is a special case of *Weil's bound* for curves over finite fields. Weil's bound for Fermat curves is easy to prove by means of Gauss and Jacobi sums, as well as its generalisation to *diagonal equations* in several variables, see [IR90], [LN83] or [Sma91]. An alternative proof is based on the character theory of a finite Frobenius group, see [Fei67, Section 26] for the basic argument and [Mata] for a refinement.

Weil's upper bound for $N_n(a, b, q)$ is not optimal when n (and with it the genus of the curve) is relatively large with respect to q . Better upper bounds in this situation were found by García and Voloch, using tools from algebraic geometry. According to [GV88, Corollary 1], rewritten here after elementary calculations, if s is an integer such that $1 \leq s \leq n-3$ and $sn \leq p$ then

$$(4) \quad N_n(a, b, q) \leq \left(\frac{s^2 - s - 2}{4} + \frac{4}{s+3} \right) n^2 + 2 \frac{n(q-1-d)}{s+3} + d,$$

where d is the number of \mathbb{F}_q -rational points of the curve with $xyz = 0$. García and Voloch pointed out that their bounds (4) hold in more general circumstances where the assumption $sn \leq p$ may not be satisfied, and described those circumstances in detail for the cases $s = 1, 2$. In particular, the case $s = 1$ of (4), which reads

$$(5) \quad N_n(a, b, q) \leq (n(n+q-1) - d(n-2))/2,$$

is valid without the assumption $n \leq p$, for p odd, except when n has the form $n = (q-1)/(r-1)$ for some subfield \mathbb{F}_r of \mathbb{F}_q (which are true exceptions). Bound (5) is better than Weil's upper bound, roughly, when n is larger than $\sqrt{q}/2$. In Corollary 14 we establish bound (5) under the assumptions that n divides $q-1$ (which is harmless in view of the next paragraph) and that $n > (q-1)/(p-1)$.

The set G of n th powers in \mathbb{F}_q^* coincides with the set of m th powers, where $m = (n, q-1)$, and is the subgroup of \mathbb{F}_q^* of order $(q-1)/m$. Setting $(\alpha, \beta) \mapsto a\alpha^n$ gives an m^2 -to-one map of the set of pairs $(\alpha, \beta) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$ with $a\alpha^n + b\beta^n = 1$ onto the set $aG \cap (1 - bG)$. Consequently, we have $N_n(a, b, q) = m^2|aG \cap (1 - bG)| + d$, where d is the number of projective \mathbb{F}_q -rational points of the curve $ax^n + by^n = z^n$ with $xyz = 0$. However, it is easy to see that d coincides with the number of projective \mathbb{F}_q -rational points of the curve $ax^m + by^m = z^m$ with $xyz = 0$, and hence $N_n(a, b, q) = N_m(a, b, q)$. Since García and Voloch's bounds (4) (as well as Weil's bound) do not increase by replacing n with m (and leaving d unchanged), it is no loss to assume that n divides $q-1$ in the sequel. In the next result we use Lemma 9 to produce an upper bound for $|aG \cap (1 - bG)|$. The first part of the argument is analogous to the proof of Theorem 2 in [Matb].

Theorem 12. *Let G be a subgroup of \mathbb{F}_q^* with $|G| < p - 1$, and let $a, b \in \mathbb{F}_q^*$. Set $e = 0, 1, 2, 3$ according as none, one, two or all three of a , b and $-a/b$ (counting repetitions) belong to G . Then $|aG \cap (1 - bG)| \leq (|G| + 1 - e)/2$.*

Proof. The elements of the cosets aG and bG of G in \mathbb{F}_q^* are the roots of the polynomials $x^k - a^k$ and $x^k - b^k$, where $k = |G|$. Consequently, the elements of $aG \cap (1 - bG)$ are the roots of the greatest common divisor $(x^k - a^k, (1 - x)^k - b^k)$, which we write in the form $(x^k - a^k)/f(x)$, where $f(x) = \prod_{\xi \in aG \setminus (1 - bG)} (x - \xi)$. Hence $|aG \cap (1 - bG)| = k - h$, where $h = \deg(f)$. There exists a polynomial $g(x) \in \mathbb{F}_q[x]$, necessarily of degree h and with leading coefficient $(-1)^k$, such that

$$(1 - x)^k - b^k = g(x) \frac{x^k - a^k}{f(x)},$$

and hence

$$(6) \quad f(x)(1 - x)^k = b^k f(x) - a^k g(x) + x^k g(x).$$

The polynomial $f(x)(1 - x)^k$ has 1 as a root with multiplicity exactly k or $k + 1$ according as $a \notin G$ or $a \in G$. According to Lemma 9, its weight is at least $k + 1$ in the former case, and at least $k + 2$ in the latter. However, the polynomial at the right-hand side of (6) has weight at most $2h + 2$. Consequently, in any case we have $2h + 2 \geq k + 1$, that is, $k - h \leq (k + 1)/2$. This is the desired conclusion in case $e = 0$.

The remaining cases are established by taking into account whether $a \in G$, and noting that the right-hand side of (6) has actually weight at most $2h + 1$ if either b or $-a/b$ belongs to G , and at most $2h$ if both do. In fact, if $b \in G$ then $g(x)$ has 0 as a root, and hence has no constant term, while if $-a/b \in G$ then $b^k f(x)$ and $a^k g(x)$ have the same leading coefficient. \square

Remark 13. We sketch a minor variation of the proof of Theorem 12 which emphasizes the connection with the linear recurrence relations for binomial coefficients discussed in the previous section. For simplicity we restrict ourselves to the case $e = 0$. Expanding the product on the left-hand side of (6) and writing $f(-x) = \sum_{j=0}^h f_j x^j$ we obtain that

$$(7) \quad f_h \binom{k}{s} + f_{h-1} \binom{k}{s+1} + \cdots + f_0 \binom{k}{s+h} = 0$$

for each integer s such that x^{s+h} has coefficient zero in the polynomial at the right-hand side of (6). This certainly holds for $1 \leq s \leq k - h - 1$, and hence the sequence of binomial coefficients $\binom{k}{i}$ modulo p , restricted to the range $1 \leq i \leq k - 1$, satisfies a linear recurrence relation of order h , with characteristic polynomial prime to $x + 1$. According to Theorem 10 we have $2h > k - 2$, and the conclusion follows.

Corollary 14. *Let q be a power of the prime p , n a divisor of $q - 1$ with $n > (q - 1)/(p - 1)$, and $a, b \in \mathbb{F}_p^*$. Then the Fermat curve $ax^n + by^n = z^n$ has at most $(n(n + p - 1) - d(n - 2))/2$ projective \mathbb{F}_p -rational points, where d is the number of points with $xyz = 0$.*

Proof. The number of projective \mathbb{F}_q -rational points of the curve with $xyz \neq 0$ equals $n^2 |aG \cap (1 - bG)|$, where G is the subgroup of \mathbb{F}_q^* of order $(q - 1)/n$. According to Theorem 12, this number is at most $(n(q - 1 + n - en))/2$. Adding to this the number of points with $xyz = 0$, which is $d = en$, we reach the conclusion. \square

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